NONSTIFFNESS OF A NONSHALLOW SPHERICAL DOME

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An asymptotic method, developed in [1 to 3], is used to investigate the nonstiffness of a nonshallow spherical dome. Vorovich [4] introduced the class of nonstiff shells, i.e., shells having for given boundary conditions, and no external loading, trivial ones equilibrium states different from the trivial ones. A characteristic property of a nonstiff shell is the fact that the lower critical loading for such shells is a negative quantity.

A rigorous proof of the existence of nonstiff shells, and a corresponding numerical analysis of the problem within the scope of shallow theory were given in [5 and 6]. Here the nonstiffness is proved strictly for a nonshallow spherical dome with fixed hinged clamping along the edge. Namely, it is shown that still another equilibrium mode, similar to the mirror image, exists in addition to the trivial mode. The nonlinear Reissner equations [7] are used for the finite symmetric deformation of thin shells of revolution, derived without assumptions on the smallness of the angle of rotation of a shell element due to deformation.

These equations contain a natural small parameter ϵ^2, the relative thinness of the walls ($\epsilon^2 = h^2/a^2\gamma^2$, where h is the thickness, a the radius of the sphere, and γ is a known number) in the higher derivatives. To prove the existence of a second solution, asymptotic expansions are first constructed for small ϵ (Section 2), and then the existence of the second solution is proved for the problem for which the constructed asymptotic expansions are valid (Section 3).

Let us note that if the formal construction of the asymptotic goes through for any spherical segments (including large hemispheres), then the proposed method for the existence of the second solution is connected to the constraint that the dome be less than a hemisphere.

1. Formulation of the problem.

Let us consider a system of nonlinear Reissner differential equations for the symmetric deformation of a spherical dome in the absence of loading

$$\varepsilon^2\left\{\frac{d}{d\xi}\left(\sin\xi\frac{du}{d\xi}\right)+\cos(\xi-u)\,\frac{\sin(\xi-u)-\sin\xi}{\sin\xi}-\right. \qquad (1.1)$$
$$\left.-\nu\left[\cos(\xi-u)-\cos\xi\right]\right\}+v\sin(\xi-u)=0$$

$$\varepsilon^2\left\{\frac{d}{d\xi}\left(\sin\xi\frac{dv}{d\xi}\right)-\left[\frac{\cos^2(\xi-u)}{\sin\xi}-\nu\left(1-\frac{du}{d\xi}\right)\sin(\xi-u)\right]v\right\}+$$
$$+\cos\xi-\cos(\xi-u)=0$$
$$(0\leqslant\xi\leqslant b<{}^1/_2\pi,\qquad 0<\nu<0.5)$$

with boundary conditions corresponding to a fixed hinge clamping of the dome along the edge

$$u(0) = 0, \qquad v(0) = 0$$

$$\left[\frac{dv}{d\xi} - \nu\frac{v\cos(\xi - u)}{\sin\xi}\right]_{\xi=b} = 0, \qquad \left[\frac{du}{d\xi} + \nu\frac{\sin\xi - \sin(\xi - u)}{\sin\xi}\right]_{\xi=b} = 0 \quad (1.2)$$

All the quantities in (1.1) and (1.2) are dimensionless. Here

$$v = \frac{\Psi}{aEh}, \quad u = \xi - \Phi, \quad \varepsilon^2 = \frac{h^2}{a^2\gamma}, \quad \gamma = 12(1 - \nu^2)$$

(see Formula (5.8) in [7]).

Here u is the angle of rotation of a shell element due to deformation, Ψ is the stress function; ν the Poisson ratio, E is Young's modulus, h the shell thickness, r the radius of the sphere, ξ a parameter corresponding to the arc length of a great circle of a unit sphere. The small parameter ϵ^2 characterizes the relative thinness of the shell wall.

It is easy to see that the problem (1.1), (1.2) has the trivial solution $v = u \equiv 0$. This solution corresponds to an equilibrium mode with zero stresses and strains. Let us consider small values of the parameter ϵ^2. Let up put $\epsilon = 0$. Eqs. (1.1) will transform into algebraic equations

$$v_0 \sin(\xi - u_0) = 0, \qquad \cos\xi - \cos(\xi - u_0) = 0 \quad (1.3)$$

Here there are two solutions. One $v_0 = u_0 \equiv 0$ is trivial, and simulataneously a solution of the problem (1.1), (1.2). The other solution

$$v_0 = 0, \ u_0 = 2\xi \quad (1.4)$$

corresponds to an equilibrium mode similar to the mirror image.

The solution (1.4) satisfies (1.1), but does not satisfy the boundary conditions at $\xi = b$ in (1.2). It is hence natural to expect the problem (1.1), (1.2) to have a second solution for small ϵ, which will behave similarly to (1.4) everywhere within the domain, but will undergo rapid changes only near the boundary that the boundary conditions (1.2) will be satisfied.

2. Construction of the asymptotic. Let us introduce some notation. Let the vector $\mathbf{V} \equiv (u, v)$ be a solution, and $\mathbf{P}[\mathbf{V}]$ the left side of the system (1.1) to (1.4). Asymptotic expansions

$$u = \sum_{s=0}^{n} \varepsilon^s u_s + \sum_{s=0}^{n} \varepsilon^s g_s + \sum_{s=0}^{n} \varepsilon^s \beta_s + z_n$$
$$v = \sum_{s=0}^{n} \varepsilon^s v_s + \sum_{s=0}^{n} \varepsilon^s h_s + \sum_{s=0}^{n} \varepsilon^s \alpha_s + x_n \quad (2.1)$$

are constructed for the second solution.

The functions $u_s(\xi)$ and $v_s(\xi)$ are obtained by using the first iteration process [8]. Namely, we demand that

$$\mathbf{P}[\mathbf{V}_n] = O(\varepsilon^{n+1}), \qquad \mathbf{V}_n \equiv \left(\sum_{s=0}^{n} \varepsilon^s u_s, \quad \sum_{s=0}^{n} \varepsilon^s v_s\right)$$

Equating the coefficients of different powers of ϵ to zero, we obtain the system (1.3) (whereby the second solution (1.4) is chosen) to determine u_0, v_0, and a system of linear homogeneous equations to determine u_s, v_s. Hence

$$u_s(\xi) = v_s(\xi) = 0 \qquad (s = 1, 2, \ldots, n)$$

Functions of boundary layer type $h_s(\xi)$, $g_s(\xi)$ are obtained by using the second iteration process [8]. To do this we seek the differences $v - v_0$ and $u - u_0$ ($v_0 = 0$, $u_0 = 2\xi$) as

$$v = \sum_{s=0}^{n} \varepsilon^s h_s, \qquad v - 2\xi = \sum_{s=0}^{n} \varepsilon^s g_s \tag{2.2}$$

Let us substitute (2.2) into (1.1), (1.2), let us make the change $\xi = b - \epsilon t$, and then let us use Taylor expansions in the neighborhood of $\epsilon = 0$ for the functions

$$\sin(b - \varepsilon t), \quad \cos(b - \varepsilon t), \quad \sin\left(b - \varepsilon t + \sum_{s=0}^{n} \varepsilon^s g_s\right), \quad \cos\left(b - \varepsilon t + \sum_{s=0}^{n} \varepsilon^s g_s\right)$$

and we equate the coefficients of ε^0, $\varepsilon^1, \ldots$, ε^n to zero. We obtain a system of nonlinear differential equations to obtain h_0, g_0:

$$\sin b\, g_0'' - h_0 \sin(b + g_0) = 0, \qquad \sin b h_0'' - \cos(b + g_0) + \cos b = 0$$

To obtain h_1, g_1 we obtain the system

$$\sin b g_1'' + h_0 (t - g_1) \cos(b + g_0) - t g_0'' \cos b - \cos b\, g_0' - - h_1 \sin(b + g_0) = 0 \tag{2.4}$$

$$\sin b h_1'' - t h_0' \cos b - h_0' \cos b + (g_1 - t) \sin(b + g_0) + t \sin b = 0$$

Analogously, we derive the first boundary condition for h_0 and g_0 at $t = 0$ from (1.2), and the second boundary condition is obtained from the requirement that the solution have boundary layer character in the neighborhood of $\xi = b$, i.e.

$$h_0'(0) = g_0'(0) = 0, \qquad h_0(\infty) = 0, \quad g_0(\infty) = 0 \tag{2.5}$$

There results from (2.3) and (2.5) that

$$h_0 = g_0 = 0 \tag{2.6}$$

Now, using (2.6) we deduce from (2.4)

$$g_1'' - h_1 = 0, \qquad h_1'' + g_1 = 0$$

with the boundary conditions

$$h_1'(0) = 0, \qquad g_1'(0) = 2(1 + \nu), \qquad g_1(\infty) = h_1(\infty) = 0$$

We hence obtain

$$h_1(\xi) = a(\xi, \varepsilon)(\cos\alpha + \sin\alpha), \qquad g_1(\xi) = a(\xi, \varepsilon)(\sin\alpha - \cos\alpha)$$

$$a(\xi, \varepsilon) = \sqrt{2}(1 + \nu)\exp\left(\frac{b - \xi}{\sqrt{2}\varepsilon}\right), \qquad \alpha = \frac{\sqrt{2}}{2}\frac{b - \xi}{\varepsilon} \tag{2.7}$$

The functions h_s, g_s ($s \geqslant 2$) are found analogously from a system of linear equations of the form (2.7), but inhomogeneous now, where the right sides are finite polynomials consisting of members of the form

$$t^m [B \sin(^1/_2 \sqrt{2}\, lt) + C \cos(^1/_2 \sqrt{2}\, nt)] \exp(-^1/_2 \sqrt{2}\, kt)$$

where m, k, l and n are integers not greater than s. It is easy to see that h_s, g_s will be functions of boundary layer type.

Finally, let us introduce the infinitely differentiable monotonous functions $\beta_s(\xi)$, $\alpha_s(\xi)$, which cancel the residual (of exponential order of smallness) in satisfying the boundary conditions (1.2) at $\xi = 0$ for the functions g_s and h_s, respectively:

$$\beta_s(\xi) = \begin{cases} -g_s(0) & (0 \leqslant \xi \leqslant 0.1b), \\ 0 & (02b \leqslant \xi \leqslant b), \end{cases} \qquad \alpha_s(\xi) = \begin{cases} -h_s(0) & (0 \leqslant \xi \leqslant 0.1b) \\ 0 & (02b \leqslant \xi \leqslant b) \end{cases}$$

Therefore, the asymptotic expansions (2.1) may be rewritten as follows:

$$u = 2\xi + \sum_{s=1}^{n} \varepsilon^s g_s + \sum_{s=1}^{n} \varepsilon^s \beta_s + z_n, \qquad v = \sum_{s=1}^{n} \varepsilon^s h_s + \sum_{s=1}^{n} \varepsilon^s \alpha_s + x_n \tag{2.8}$$

The notation

$$\psi_n = v - x_n, \qquad \varphi_n = u - z_n \tag{2.9}$$

will be used later in Section 3.

Let us note that the estimates*

$$|\varphi_n| \leqslant m_1 \varepsilon \xi, \qquad |\psi_n| \leqslant m_2 \varepsilon \xi \tag{2.10}$$

are easily established from (2.8) and the explicit expressions for h_s, g_s, α_s and β_s.

3. Foundation of the asymptotic. Existence of a nontrivial solution. Let us introduce the following spaces of the vectors $\mathbf{V} \equiv (u, v)$.

1) A space consisting of vectors with the finite norm

$$(X) \qquad \|\mathbf{V}\|_X = \|u\|_{C_2} + \|v\|_{C_2}$$

where C_2 denotes the Banach space whose elements are all twice continuously differentiable functions in the segment $[0, b]$, which vanish for $\xi = 0$;

2) A space of pairs $\lambda = (\mathbf{f}, \varphi)$, where $\mathbf{f} \equiv (f_1, f_2)$, $\varphi \equiv (\varphi_1, \varphi_2)$, i.e., the space of quadruples $\lambda \equiv (f_1, f_2, \varphi_1, \varphi_2)$ with the norm

$$(Y) \qquad \|\lambda\|_Y = \|f_1\|_{C_0} + \|f_2\|_{C_0} + |\varphi_1(b)| + |\varphi_2(b)|$$

where C_0 denotes the Banach space of all continuous functions with the finite norm

$$\|f_1\|_{C_0} = \max |f_1 / \xi| \qquad (0 \leqslant \xi \leqslant b)$$

We shall consider the problem (1.1), (1.2) as a functional equation

$$\mathbf{P}(\mathbf{V}) = 0 \tag{3.1}$$

where the operator $\mathbf{P}$ is defined by the left side of the system (1.1), (1.2). Let us show that the operator $\mathbf{P}$ acts from the space X into Y. To do this, we note that the relationships

$$u(\xi) = u'(0)\,\xi + u''(\xi_1)\,\xi^2 \quad (0 \leqslant \xi_1 \leqslant b), \qquad |u(\xi)| \leqslant \xi \|u\|_{C_2}$$

are valid for any function u from the space C_2.

Let us now consider the first Eq. in (1.1). Using the fact that u and v are elements from C_2, we easily deduce the inequalities

$$|\cos(\xi - u) - \cos\xi| = 2\,|\sin(\xi - {}^1/_2 u)\sin{}^1/_2 u| \leqslant |u(\xi)| \leqslant \xi \|u\|_{C_2}$$

$$|\sin \xi u''| \leqslant \xi \|u\|_{C_2} \quad (0 \leqslant \xi_i \leqslant b) \tag{3.2}$$

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* Here and henceforth throughout, m_i are some positive constants not dependent on ξ and ϵ.

$$\left|\cos\xi u' + \frac{1}{2}\frac{\sin 2(\xi-u)}{\sin\xi} - \cos(\xi-u)\right| = \left|\cos\xi u'(0) + \cos\xi u''(\xi_2)\xi + \right.$$

$$\left. + \frac{1}{2}\frac{\sin 2(\xi-u)}{\sin\xi} - \cos\xi + (\cos\xi - \cos(\xi-u))\right| \leqslant \left|(u'(0)-1)\cos\xi + \right.$$

$$\left. + \frac{1}{2}\frac{\sin 2(\xi-u)}{\sin\xi}\right| + |u''(\xi_3)\xi\cos\xi + (\cos\xi - \cos(\xi-u))| \leqslant$$

$$\leqslant \left|(u'(0)-1)2\sin^2\frac{\xi}{2} + \xi r(\xi_4)\right| + m_3\xi\|u\|_{C_2} \leqslant m_4\xi\|u\|_{C_2}$$

since

$$\frac{1}{2}\frac{\sin 2(\xi-u)}{\sin\xi} = 1 - u'(0) + \xi r(\xi_4), \quad |r(\xi_4)| \leqslant \max\left[\frac{\sin 2(\xi-u)}{\sin\xi}\right]'(0 \leqslant \xi \leqslant b)$$

Applying the inequalities (3.2) and analogous estimates to the second equation in (1.1) and the boundary conditions (1.2), we obtain that the operator **P** acts from X into Y.

Theorem 3.1. Besides the trivial solution $u = v \equiv 0$, the problem (1.1), (1.2) has another solution for which the asymptotic expansions (2.10) are valid, where the following estimates hold:

$$\max |x_n(\xi)| \leqslant m_3\varepsilon^n, \qquad \max |z_n(\xi)| \leqslant m_3\varepsilon^n \qquad (0 \leqslant \xi \leqslant b) \quad (3.3)$$

For the proof we use a theorem [2] which permits establishment of the existence of a solution in the neighborhood of $\mathbf{V}_k^*$, where a segment of the asymptotic series is taken as $\mathbf{V}_k^* \equiv (\varphi_k, \psi_k)$.

Theorem 3.2. Let the operator **P** be defined in the sphere Ω ($\|\mathbf{V} - \mathbf{V}_k^*\| \leqslant R$) of the space X, and have a continuous second derivative in the sphere Ω_0 ($\|\mathbf{V} - \mathbf{V}_k^*\| \leqslant r < R$) . Moreover, let there exist an operator

$$\Gamma_\varepsilon(\mathbf{V}) = [\mathbf{P}'_{\mathbf{V}_k^*}(\mathbf{V})]^{-1}$$

and let be satisfied the conditions

$$(1) \quad \|\mathbf{P}(\mathbf{V}_k^*)\|_Y \leqslant m_1\varepsilon^{k+1} \qquad (2) \quad \|\mathbf{P}_\mathbf{V}''\| \leqslant m_3$$

$$(3) \quad \|\Gamma_\varepsilon\|_{(Y\to X)} \leqslant m_2\varepsilon^{-m} \qquad (2m < k+1) \qquad (3.4)$$

Then (3.1) has the solution $\mathbf{V}^*$ for sufficiently small ϵ:

$$\varepsilon < (2m_1m_2^2m_3)^{2m-k-1}$$

and the following estimate is valid

$$\|\mathbf{V}^* - \mathbf{V}_k^*\|_X \leqslant C\varepsilon^{k+1-m}$$

Proof. Let us show that the conditions of Theorem 3.2 are satisfied, where $m = 4$ is independent of k, and k can be chosen such that $k > 2m - 1$.

The first estimate of (3.4) can be established directly from the relationship

$$\mathbf{P}(\mathbf{V}_k^*) = O(\varepsilon^{k+1}), \ \mathbf{V}_k^* \equiv (\varphi_k, \psi_k) \qquad (3.5)$$

which is easily verified by substituting ψ_k and ϕ_k in the left side of the system (1.1), (1.2).

Furthermore, we show that the following estimate holds

$$\|\Gamma_\varepsilon\|_{(Y\to X)} \leqslant m_2\varepsilon^{-4} \qquad (3.6)$$

To do this we consider the Frechet derivative on the element $\mathbf{V}_k^*$:

$$P_{V_k^*}(V) \equiv \Big\{\varepsilon^2\Big[\sin\xi u'' + \cos\xi u' - \frac{\cos 2(\xi-\varphi_k)}{\sin\xi}u - (1+\nu)u\sin(\xi-\varphi_k)\Big] - $$
$$-\psi_k u\cos(\xi-\varphi_k) + v\sin(\xi-\varphi_k)$$

$$\varepsilon^2\Big(\sin\xi v'' + \cos\xi v' - \Big[\frac{\cos^2(\xi-\varphi_k)}{\sin\xi} - \nu(1-\varphi_k')\sin(\xi-\varphi_k)\Big]v - $$
$$-\psi_k\Big[\frac{\sin 2(\xi-\varphi_k)}{\sin\xi}u + \nu u'\sin(\xi-\varphi_k) + \nu u(1-\varphi_k')\cos(\xi-\varphi_k)\Big] - \sin(\xi-\varphi_k)u\Big\}$$

and for $\zeta = b$:

$$\Big\{u' + \nu\frac{u}{\sin\xi}\cos(\xi-\varphi_k),\quad v' - \nu v\frac{\cos(\xi-\varphi_k)}{\sin\xi} - \nu\psi_k u\frac{\sin(\xi-\varphi_k)}{\sin\xi}\Big\}$$

Let us consider the system of equations

$$P'_{V_k^*}(V) = \mathbf{f}, \qquad \mathbf{f} \equiv (f_1, f_2) \tag{3.7}$$

Utilizing (2.8) and (2.9), we rewrite (3.7) as

$$\varepsilon^2\Big\{\sin\xi u'' + \cos\xi u' - u\frac{\cos 2(\xi+\varepsilon s_2)}{\sin\xi} + (1+\nu)u\sin(\xi+\varepsilon s_2)\Big\} -$$
$$- v\sin(\xi+\varepsilon s_2) - \varepsilon s_1 u\cos(\xi+\varepsilon s_2) = f_1$$

$$\varepsilon^2\Big\{\sin\xi v'' + \cos\xi v' - \Big[\frac{\cos^2(\xi+\varepsilon s_2)}{\sin\xi} - \nu(1+\varepsilon s_2')\sin(\xi+\varepsilon s_2)\Big]v +$$
$$+\varepsilon s_1 u\frac{\sin 2(\xi+\varepsilon s_2)}{\sin\xi} + \nu\varepsilon u' s_1\sin(\xi+\varepsilon s_2) + \nu\varepsilon s_1 u(1+\varepsilon s_2')\cos(\xi+\varepsilon s_2)\Big\} +$$
$$+ u\sin(\xi+\varepsilon s_2) = f_2 \tag{3.8}$$

with the boundary conditions

$$u = v = 0 \quad \text{for} \quad \xi = 0$$
$$u' + \nu\frac{u}{\sin\xi}\cos(\xi+\varepsilon s_2) = \varphi_1 \quad \text{for} \quad \xi = b \tag{3.9}$$
$$v' - \nu\frac{v\cos(\xi+\varepsilon s_2)}{\sin\xi} + \nu\varepsilon s_1 u\frac{\sin(\xi+\varepsilon s_2)}{\sin\xi} = \varphi_2 \quad \text{for} \quad \xi = b$$
$$s_1 = \varepsilon^{-1}\psi_k, \qquad s_2 = \varepsilon^{-1}(\varphi_k - 2\xi)$$

We shall henceforth consider that $\varphi_1 = \varphi_2 = 0$ for $\zeta = b$ since reduction to this case is executed simply by replacing u and v, respectively, by

$$u + \frac{\varphi_1\xi}{1+\nu k_1 b}, \qquad v + \frac{1}{1-\nu k_1 b}\Big(\varphi_2 - \frac{\varepsilon k_2 b\varphi_1}{1+\nu k_1 b}\Big)\xi$$

where k_1 and k_2 are the numbers

$$k_1 = \frac{\cos(b+\varepsilon s_2(b))}{\sin b}, \qquad k_2 = \frac{\nu s_1(b)\sin(b+\varepsilon s_2(b))}{\sin b}$$

where we have $k_1 > 0$ for $b < \frac{1}{2}\pi$ and sufficiently small ϵ. The boundary conditions will be homogeneous with such a substitution, and f_1 and f_2 on the right sides acquire terms of the form

$$m_1\varphi_1(b)\xi + m_2\varphi_2(b)\xi.$$

Let us multiply the first equation of (3.6) by $v + u$, the second by $v - u$, and let us integrate between 0 and b and then add. We hence obtain

$$\varepsilon^2\Big[\int_0^b \sin\xi u'^2 d\xi+\int_0^b \sin\xi v'^2 d\xi+\int_0^b \frac{\cos^2\xi\cos 2\varepsilon s_2}{\sin\xi}u^2 d\xi+\int_0^b \frac{\cos^2(\xi+\varepsilon s_2)}{\sin\xi}v^2 d\xi\Big]_1+$$

$$+\varepsilon^2\Big[-\int_0^b u^2\sin\xi\cos 2\varepsilon s_2 d\xi-2\int_0^b u^2\cos\xi\sin\varepsilon s_2 d\xi-(1+\nu)\int_0^b u^2\sin(\xi+\varepsilon s_2)\,d\xi-$$

$$-\nu\int_0^b v^2(1+\varepsilon s_2')\sin(\xi+\varepsilon s_2)\,d\xi-\int_0^b\frac{\sin^2(\xi+\varepsilon s_2)}{\sin\xi}uv d\xi-$$

$$-(1+\nu)\int_0^b vu\sin(\xi+\varepsilon s_2)\,d\xi+$$

$$+\nu\int_0^b vu(1-\varepsilon s_2')\sin(\xi+\varepsilon S_2)\,d\xi\Big]_2-\varepsilon^3\Big[\int_0^b\frac{\sin 2(\xi+\varepsilon s_2)}{\sin\xi}s_1uv d\xi+$$

$$+\nu\int_0^b s_1vu(1+\varepsilon s_2')\cos(\xi+\varepsilon s_2)\,d\xi+\nu\int_0^b s_1u'v\sin(\xi+\varepsilon s_2)\,d\xi- \tag{3.10}$$

$$-2\int_0^b s_1u^2\cos\xi\cos\varepsilon s_2 d\xi-\int_0^b\frac{\cos 2\xi\sin\varepsilon s_2}{\sin\xi}s_1u^2 d\xi-\nu\int_0^b s_1(1+\varepsilon s_2')u^2\cos(\xi+\varepsilon s_2)\,d\xi-$$

$$-\nu\int_0^b s_1uu'\sin(\xi+\varepsilon s_2)\,d\xi\Big]_3+\int_0^b(v^2+u^2)\sin(\xi+\varepsilon s_2)\,d\xi+\varepsilon\int_0^b s_1u^2\cos(\xi+\varepsilon s_2)\,d\xi+$$

$$+\varepsilon\int_0^b s_1vu\cos(\xi+\varepsilon s_2)\,d\xi+\varepsilon^2\,[-\nu v^2(b)\cos(b+\varepsilon s_2(b))+\nu u^2(b)\cos(b+\varepsilon s_2(b))+$$

$$+2\nu v(b)\,u(b)\cos(b+\varepsilon s_2(b))]_4+\varepsilon^3\,[\nu v(b)\,u(b)\sin(b+\varepsilon s_2(b))-$$

$$-\nu s_1(b)\,u^2(b)\sin(b+\varepsilon s_2(b))]_5=\int_0^b f_1(v+u)\,d\xi+\int_0^b f_2(v-u)\,d\xi$$

(The numbers indicated after the square brackets are ordered for convenience).

Let us note that in deriving (3.10) we used an equality which is valid for any smooth function satisfying conditions (3.9) for $\varphi_1=\varphi_2=0$:

$$\int_0^b v(\sin\xi u')'d\xi-\int_0^b u(\sin\xi v')'d\xi=-2\nu v(b)\,u(b)\cos(b+\varepsilon s_2(b))+$$

$$+\varepsilon\nu s_1(b)\,u^2(b)\sin(b+\varepsilon s_2(b))$$

Let us show that the expressions in the second and third square brackets can be estimated by utilizing the inequalities

$$\varepsilon^2[\ldots\ldots]_2\leqslant m_1\varepsilon^2\int_0^b(v^2+u^2)\sin\xi d\xi \tag{3.11}$$

$$\varepsilon^3[\ldots\ldots]_3\leqslant m_2\varepsilon^3\Big(\int_0^b(v^2+u^2)\sin\xi d\xi+\int_0^b u'^2\sin\xi d\xi\Big) \tag{3.12}$$

For the proof let us note the following inequalities which are valid under the conditions that $0 < \xi < b < {}^1/_2\pi$ and ϵ is sufficiently small

$$
\begin{array}{ll}
|s_1| < m\zeta \leqslant m\,{}^1/_2\pi\sin\zeta, & |s_2| < m\xi,\ |\varepsilon s_2'| < m \\
{}^1/_2\pi\sin\zeta < \sin{}^1/_2\zeta, & \sin 2\xi < 2\sin\xi \\
|\sin\varepsilon s_2| \leqslant \sin\xi, & \cos\varepsilon s_2 > 1-\alpha,\ \ 0 \leqslant \xi+\varepsilon s_2 < {}^1/_2\pi \\
\sin(\zeta+\varepsilon s_2) \geqslant \pi^{-1}\sin\xi, & |\sin(\xi+\varepsilon s_2)| \leqslant 2\sin\zeta
\end{array}
\tag{3.13}
$$

where α can be made arbitrarily small because of the selection of small ϵ.

Utilizing the estimates (3.13) as well as arithmetic mean inequalities, we arrive at (3.11) and (3.12).

Analogously we establish the following estimates:

$$
\begin{gathered}
\int_0^b (v^2+u^2)\sin(\zeta+\varepsilon s_2)\,d\xi \geqslant \frac{1}{\pi}\int_0^b (v^2+u^2)\sin\xi d\xi \\
\varepsilon\int_0^b s_1(u^2+vu)\cos(\xi+\varepsilon s_2)\,d\zeta \leqslant m\varepsilon\int_0^b (v^2+u^2)\sin\xi d\xi
\end{gathered}
\tag{3.14}
$$

$$
\varepsilon^2[\ldots]_4 \geqslant -2\varepsilon^2\nu u^2(b)\cos(b+\varepsilon s_2(b)), \qquad \varepsilon^3[\ldots]_5 \geqslant -\varepsilon^3 m\,(u^2(b)+v^2(b))
$$

Applying (3.11) to (3.14), we deduce from (3.10)

$$
\begin{gathered}
\varepsilon^2\left[\int_0^b u'^2\sin\xi d\xi + \int_0^b \frac{\cos^2\xi\cos 2\varepsilon s_2}{\sin\xi}u^2 d\xi + \int_0^b v'^2\sin\xi d\xi + \int_0^b \frac{\cos^2(\xi+\varepsilon S_2)}{\sin\xi}v^2 d\xi\right]_1 - \\
-\varepsilon^2 m_1\int_0^b (v^2+u^2)\sin\xi d\xi - \varepsilon^3 m_2\int_0^b (v^2+u^2)\sin\xi d\xi - \varepsilon^3 m_3\int_0^b u'^2\sin\xi d\xi + \\
+\frac{1}{\pi}\int_0^b (v^2+u^2)\sin\xi d\xi - \varepsilon m_4\int_0^b (v^2+u^2)\sin\xi d\zeta - 2\varepsilon^2\nu u^2(b)\cos(b+\varepsilon s_2(b)) - \\
-\varepsilon^3\nu m_5(u^2(b)+v^2(b))\sin(b+\varepsilon s_2(b)) \leqslant \int_0^b (|f_1||v+u|+|f_2||v-u|)\,d\xi
\end{gathered}
\tag{3.15}
$$

Let us now choose ϵ so small that the inequalities

$$
\begin{gathered}
\varepsilon m_4 + \varepsilon^2 m_1 + \varepsilon^3 m_2 < \pi^{-1}, \qquad \varepsilon^3 m_3 < {}^1/_3\alpha,\ \ \cos 2\varepsilon s_2 > 1 - {}^1/_3\alpha \\
(1-\alpha)\cos b > 2\nu\cos(b+\varepsilon s_2(b)) \qquad (0<\nu<0.5) \\
\varepsilon m_5\nu\,\mathrm{tg}\,b < {}^1/_3\alpha, \qquad \cos\xi \geqslant \cos b \qquad (0 \leqslant \xi \leqslant b < {}^1/_2\pi)
\end{gathered}
\tag{3.16}
$$

would be satisfied simultaneously.

Using (3.16) and inequalities of the form

$$
u^2(b) = 2\int_2^b uu'd\xi < \int_0^b u'^2\sin\xi d\xi + \int_0^b \frac{u^2}{\sin\xi}\,d\zeta
\tag{3.17}
$$

we obtain from (3.15)

$$\frac{a}{3}\varepsilon^2\left(\int_0^b u'^2\sin\xi d\xi+\cos^2 b\int_0^b\frac{u^2}{\sin\xi}d\xi+\int_0^b v'^2\sin\xi d\xi+\cos^2 b\int_0^b\frac{v^2}{\sin\xi}d\xi\right)\leqslant$$

$$\leqslant\int_0^b(|f_1||v+u|+|f_2||v-u|)\,d\xi \qquad (3.18)$$

Applying the estimate (3.17) once again to the left side of (3.18), and the triangle inequality to the right, we deduce

$$^1/_3\alpha\varepsilon^2(\max|v^2|+\max|u^2|)\leqslant\|\mathbf{f}\|_Y(\max|v|^2+\max|u|^2)^{1/2}\qquad(0\leqslant\xi\leqslant b)$$

Hence

$$\max|v|+\max|u|\leqslant m\varepsilon^{-2}\|\mathbf{f}\|_Y\qquad(0\leqslant\xi\leqslant b)\qquad(3.19)$$

In order to obtain an estimate for the higher derivatives, let us add the term $-u\csc\xi$ to the left and right sides of the first Eq. in (3.8), and $-v\csc\xi$ to the second Eq. Then (3.8) can be rewritten as

$$\varepsilon^2\sin\xi\frac{d}{d\zeta}\frac{1}{\sin\xi}\frac{d}{d\xi}\sin\xi u=F_1,\qquad\varepsilon^2\sin\xi\frac{d}{d\zeta}\frac{1}{\sin\xi}\frac{d}{d\xi}\sin\xi v=F_2\qquad(3.20)$$

Here

$$F_1=\frac{1}{\varepsilon^2}f_1+\frac{\cos2(\xi+\varepsilon s_2)-1}{\sin\xi}-(1+\nu)u\sin(\xi+\varepsilon s_2)+\frac{1}{\varepsilon^2}v\sin(\xi+\varepsilon s_2)+$$
$$+\frac{1}{\varepsilon}s_1u\cos(\xi+\varepsilon s_2)$$

$$F_2=\frac{1}{\varepsilon^2}f_2+\frac{\cos^2(\xi+\varepsilon s_2)-1}{\sin\xi}v-\nu v(1+\varepsilon s_2')\sin(\xi+\varepsilon s_2)-\varepsilon s_1u\frac{\sin2(\xi+\varepsilon s_2)}{\sin\xi}-$$
$$-\nu\varepsilon u's_1\sin(\xi+\varepsilon s_2)-\nu\varepsilon s_1u(1+\varepsilon s_2')\cos(\xi+\varepsilon s_2)-\frac{u}{\varepsilon^2}\sin(\xi+\varepsilon s_2)$$

We now pass from (3.20) to a system of two equivalent integral equations, from which we obtain estimates of u'' and v'' by utilizing the estimates (3.19). As illustration, let us obtain estimates of u' and u''. Taking account of the boundary conditions (3.9), we have from (3.20)

$$u=\frac{1}{\sin\xi}\Phi(\xi,\ t,\ b)+k\,\mathrm{tg}\frac{\xi}{2}\Phi(b,\ t,\ b)\qquad(3.21)$$

$$\Phi(\zeta,\ t,\ b)=\int_0^\xi\sin\ t\ dt\int_b^t\frac{F_1(s)}{\sin s}ds,\qquad k=\frac{\cos b-\nu\cos(b+\varepsilon s_2(b))}{\sin b[\sin b-(\cos b-\nu\cos(b+\varepsilon s_2(b))\,\mathrm{tg}\,^1/_2 b]}$$

Furthermore, let us note the following valid estimate for F_1

$$|F_1|\leqslant m\varepsilon^{-4}\xi\|\mathbf{f}\|_{C_0}\leqslant{}^1/_2m\pi\varepsilon^{-4}\|\mathbf{f}\|_{C_0}\sin\xi\qquad(0\leqslant\xi\leqslant b<{}^1/_2\pi)\qquad(3.22)$$

This follows from (3.20) by virtue of the estimates (3.13) and the fact that $f_1\in C_0$. From (3.21) we have

$$u'=-\frac{\cos\xi}{\sin^2\xi}\Phi(\xi,\ t,\ b)+\int_b^\zeta\frac{F(s)}{\sin s}ds+\frac{k}{2\cos^2(\xi/2)}\Phi(b,t,\ b)\qquad(3.23)$$

Utilizing (3.22) and the triangle inequality, we deduce from (3.23)

$$\max|u'(\xi)|\leqslant m\varepsilon^{-4}\|\mathbf{f}|_Y\qquad(0\leqslant\xi\leqslant b)\qquad(3.24)$$

Let us consider u'':

$$u'' = \frac{1+\cos^2\xi}{\sin^3\xi}\Phi(\xi,t,b) - \frac{\operatorname{ctg}\xi}{2\sin^2(\xi/2)}\Phi(\xi,\xi,b) + \frac{F_1(\xi)}{\sin\xi} + \frac{k\sin(\xi/2)}{\cos^3(\xi/2)}\Phi(b,t,b)$$

Applying the identity

$$a_1b_1 - a_2b_2 = a_1(b_1 - b_2) + b_2(a_1 - a_2)$$

to the difference between the first two members on the right side, we have

$$u'' = -\frac{1+\cos^2\xi}{\sin^3\xi}\Phi(\xi,\xi,t) + \frac{1+\cos^2\xi}{2\sin\xi\cos^2(\xi/2)}\Phi(\xi,\xi,b) + \frac{F_1(\xi)}{\sin\xi} + \frac{k\sin\xi}{\cos^3(\xi/2)}\Phi(b,t,b) \quad (3.25)$$

Using (3.22), the triangle inequality, and also the inequalities

$$|\xi - t| < \xi, \quad 0 \leqslant \xi \leqslant b < {}^1\!/_2\pi$$

we obtain from (3.25)

$$\max|u''(\xi)| \leqslant m\varepsilon^{-4}\|\mathbf{f}\|_Y \quad (0 \leqslant \xi \leqslant b) \quad (3.26)$$

From (3.19), (3.24) and (3.26) follows:

$$\|\mathbf{V}\|_X \leqslant m_1\varepsilon^{-4}\|\mathbf{f}\|_Y, \qquad \|\mathbf{V}\|_X \leqslant m_1\varepsilon^{-4}\|\mathbf{P}_{\mathbf{V}_k^*}'(\mathbf{V})\|_Y$$

We hence easily deduce that the operator $\mathbf{P}_{\mathbf{V}_k^*}'$ on the right side of the last inequality is an inverse and the estimate (3.6) holds.

The estimate of $\|\mathbf{P}_{\mathbf{V}}''\|$ follows from an examination of the bilinear form

$$\mathbf{P}_{\mathbf{V}}''(\mathbf{V}_1)(\mathbf{V}_2), \qquad \mathbf{V}_1 \equiv (u_1, v_1), \qquad \mathbf{V}_2 \equiv (u_2, v_2) \qquad \left(\frac{\sin 2(\xi - u)}{\sin\xi}u_1u_2\right)$$

For example, a typical member of this form is exhibited in the parentheses. Evidently

$$\left|\frac{\sin 2(\xi - u)}{\sin\xi}u_1u_2\right| \leqslant m\,|\sin 2(\xi - u)|\left|\frac{u_1}{\xi^{1/2}}\right|\left|\frac{u_2}{\xi^{1/2}}\right| \leqslant m\|u_1\|_{C_2}\|u_2\|_{C_2}$$

In the general case we obtain

$$\|\mathbf{P}_{\mathbf{V}}''(\mathbf{V}_1)(\mathbf{V}_2)\|_Y \leqslant m_3\|\mathbf{V}_1\|_X\|\mathbf{V}_2\|_X$$

from which the second estimate in (3.4) indeed follows.

Thus the conditions of Theorem 3.2 are satisfied if $k > 7$ and ϵ is sufficiently small $(0 < \varepsilon < \varepsilon_1)$. Hence, (3.1) has the solution $\mathbf{V}^* \equiv (u, v)$, for which the estimate

$$\|\mathbf{V}^* - \mathbf{V}_k^*\| \leqslant m\varepsilon^{\lambda-3} \qquad (k > 7) \quad (3.27)$$

is valid.

Now applying the triangle inequality and explicit expressions for the functions h_s, g_s, we obtain estimates of x_n, z_n and their derivatives from (3.27).

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